数学分析 标准答案

一. 计算:

1. $\lim\limits_{n\to\infty}\left(1+\frac{1}{2n}-\frac{1}{n^2}\right)^n$

$$= e^{\lim\limits_{n\to\infty} n\cdot\ln\left(1+\frac{1}{2n}-\frac{1}{n^2}\right)} = e^{\frac{1}{2}}$$

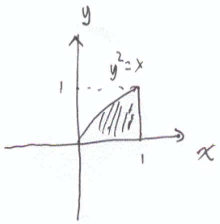
2. $\lim\limits_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{x\tan x}\right)$

$$= \lim_{x\to 0}\frac{\tan x - x}{x^2\tan x} = \lim_{x\to 0}\frac{\tan x - x}{x^3} = \lim_{x\to 0}\frac{\frac{1}{\cos^2 x}-1}{3x^2}$$

$$= \lim_{x\to 0}\frac{1-\cos^2 x}{3x^2}\cdot\frac{1}{\cos^2 x} = \lim_{x\to 0}\frac{2\cos x\sin x}{6x} = \frac{1}{3}$$

3. $\int x\sqrt{1-x}\,dx = \int (x-1+1)\sqrt{1-x}\,dx = \int\sqrt{1-x}\,dx - \int (1-x)^{\frac{3}{2}}dx$

4. $\int_0^1 f(x)\,dx = \int_0^1 dx\int_0^{\sqrt{x}} e^{-\frac{y^2}{2}}dy$



$$= \int_0^1 dy\int_{y^2}^1 e^{-\frac{y^2}{2}}dx$$

$$= \int_0^1 e^{-\frac{y^2}{2}}(1-y^2)\,dy = \int_0^1 e^{-\frac{y^2}{2}}dy - \int_0^1 e^{-\frac{y^2}{2}}y^2\,dy$$

而 $\int_0^1 e^{-\frac{y^2}{2}}y^2\,dy = -\int_0^1 \cancel{e^{\frac{y^2}{2}}}\,y\,d\,e^{-\frac{y^2}{2}} = -ye^{-\frac{y^2}{2}}\Big|_0^1 + \int_0^1 e^{-\frac{y^2}{2}}dy$

$$= -e^{-\frac{1}{2}}$$

$$e^{f(y)} + x e^{f(y)} \cdot f'(y) \cdot y'_x = e^{y} \cdot y'_x \qquad \therefore\ y'_x = \frac{e^{f(y)}}{e^{y} - x e^{f(y)} \cdot f'(y)}$$

$$= \frac{1}{x(1-f'(y))}$$

$$\therefore\ y''_x = \frac{-1}{x^2(1-f'(y))^2} \cdot [1-f'(y) + x(-f''(y)\cdot y'_x)]$$

$$= \frac{f''(y) - (1-f'(y))^2}{x^2(1-f'(y))^3}$$

6. $z_y = 3x(1-2xy)^{3x-1} \cdot (-2x)$ $\qquad \ln z = 3x \ln(1-2xy)$

$$\frac{1}{z} z'_x = 3\ln(1-2xy) + 3x \cdot \frac{-2y}{1-2xy}$$

$$dz = z_x dx + z_y dy$$

7. $I = \iint_D \frac{\partial}{\partial x}[x^2-4y] - \frac{\partial}{\partial y}[2xy-2y]\, dxdy$ $\qquad D: x^2+y^2 \le 9$

$$= \iint_D (2x - 2x + 2)\, dxdy = 2\iint_D dxdy = 2 \cdot \pi \cdot 9$$

8. 原式 $= \iiint_\Omega [(x^2+y^2+z^2) + 1 - 2x]\, dxdydz$

$$= \iiint_\Omega (x^2+y^2+z^2)\, dv + \iiint_\Omega 1\, dv - 0$$

$$= \frac{4}{3}\pi \cdot 1^3 + \int_0^{2\pi} d\theta \int_0^{\pi} d\varphi \int_0^1 r^2 \cdot r \sin\varphi\, dr$$

$= \frac{4}{3}\pi + 2\pi \cdot (-\cos\varphi)\Big|_0^{\pi} \cdot \frac{1}{[illegible]} = \frac{8}{3}\pi$.

证，$f(x) = \sin x + \sin^2 x + \cdots + \sin^n x - 1$

$f'(x) = \cos x + 2\sin x\cos x + \cdots + n\sin^{n-1} x \cos x > 0$ $\quad x \in (\frac{\pi}{6}, \frac{\pi}{2})$

$\therefore f(x) \uparrow$

而 $f(\frac{\pi}{2}) = n-1 > 0 \quad (n>1) \quad f(\frac{\pi}{6}) = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} - 1 < 0$

$\therefore$ 由零点定理，$f(x)=0$ 在 $(\frac{\pi}{6}, \frac{\pi}{2})$ 内有唯一根 x_n

由 $\sin x + \cdots + \sin^n x = 1$

得 $\dfrac{\sin x - \sin^{n+1} x}{1-\sin x} = 1 \quad \therefore 2\sin x - \sin^{n+1} x = 1$

$\sin x = \frac{1}{2}(1+\sin^{n+1} x)$ $\quad$ ~~$\frac{1}{2} < \sin x < 1 \quad x \in (\frac{\pi}{6}, \frac{\pi}{2})$~~

$\therefore$ ~~$\sin x$~~ $\quad$ 令 $y = \sin x \quad \frac{1}{2} < y < 1$

有 $y = \frac{1}{2}(1+y^{n+1}) \qquad y_n = \frac{1}{2}(1+y_n^{n+1})$

$y_n + y_n^2 + \cdots + y_n^n = 1 \qquad \frac{1}{2} < y_n < 1$

下面证 y_n 递减，反证，若有某个 n_0 使 $y_{n_0} < y_{n_0+1}$

此时，$1 = y_{n_0} + \cdots + y_{n_0}^{n_0} < y_{n_0+1} + \cdots + y_{n_0+1}^{n_0} \le y_{n_0+1} + \cdots + y_{n_0+1}^{n_0} + y_{n_0+1}^{n_0+1} = 1$ 矛盾。

$\therefore y_n$ 递减有下界，$\therefore \lim_{n\to\infty} y_n$ 存在。

$0 \le y_n^n \le y_1^n \to 0. \quad \therefore \lim_{n\to\infty} y_n^n = 0.$

由 $y_n = \frac{1}{2}(1+y_n^{n+1})$ 二边取极限，$\lim_{n\to\infty} y_n = 1/2$

$\therefore y_n = \sin x_n \qquad x_n = \arcsin y_n \to \frac{\pi}{\ }$

一. 1. $\varphi(x) = \int_0^1 f(xt)\,dt$

$$\overset{xt=u}{=} \int_0^x f(u)\cdot\frac{1}{x}\,du \qquad x\neq 0$$

$\therefore$ $x\neq 0$时 $\varphi'(x) = \frac{f(x)}{x} - \frac{1}{x^2}\int_0^x f(u)\,du$

$$\varphi'(0) = \lim_{x\to 0}\frac{\varphi(x)-\varphi(0)}{x-0} = \lim_{x\to 0}\frac{\int_0^1 f(xt)\,dt - 0}{x}$$

$$= \lim_{x\to 0}\frac{\frac{1}{x}\int_0^x f(u)\,du}{x} = \lim_{x\to 0}\frac{\int_0^x f(u)\,dx}{x^2} = \lim_{x\to 0}\frac{f(x)}{2x} = \frac{1}{2}$$

$$\therefore \varphi'(x) = \begin{cases} \frac{1}{2} & x=0 \\ \frac{f(x)}{x} - \frac{1}{x^2}\int_0^x f(u)\,du & x\neq 0 \end{cases}$$

$$\lim_{x\to 0}\varphi'(x) = \lim_{x\to 0}\left[\frac{f(x)}{x} - \frac{1}{x^2}\int_0^x f(u)\,du\right] = \frac{1}{2}-\frac{1}{2} = 0.$$

$\varphi'(x)$在$x=0$处不连续.

二. 1. $a_n = \int_0^1 \frac{t^n}{1+t^2}\,dt$

$$\frac{1}{n}(a_n + a_{n+2}) = \frac{1}{n}\int_0^1 \frac{t^n + t^{n+2}}{1+t^2}\,dt = \frac{1}{n}\int_0^1 t^n\,dt$$

$$= \frac{1}{n}\cdot\frac{1}{n+1}$$

$$\therefore \sum_{n=1}^{\infty}\frac{1}{n(n+1)} = \lim_{n\to\infty}\left[1-\frac{1}{2}+\frac{1}{2}-\frac{1}{3}+\cdots+\frac{1}{n}-\frac{1}{n+1}\right] = \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right) = 1$$

2. $0 < \frac{a_n}{n^\lambda} = \frac{1}{n^\lambda}\int_0^1 \frac{t^n}{1+t}\,dt \le \frac{1}{n^\lambda}\int_0^1 t^n\,dt = \frac{1}{n^\lambda(n+1)} \le \frac{1}{n^{\lambda+1}}$

5.

$$\frac{d}{dt}f(tx,ty) = xf_1' + yf_2'$$

$$\frac{d^2}{dt^2}f(tx,ty) = x^2f_{11}'' + 2xyf_{12}'' + y^2f_{22}''$$

$$\therefore \int_0^1 (1-t)\left[x^2f_{11}'' + 2xyf_{12}'' + y^2f_{22}''\right]dt$$

$$= \int_0^1 (1-t)\frac{d^2}{dt^2}f(tx,ty)\,dt = \int_0^1 (1-t)\,d\,\frac{d}{dt}f(tx,ty)$$

$$= (1-t)\frac{d}{dt}f(tx,ty)\Big|_0^1 - \int_0^1 \frac{d}{dt}f(tx,ty)\,d(1-t)$$

$$= 0 + \int_0^1 \frac{d}{dt}f(tx,ty)\,dt = f(tx,ty)\Big|_0^1 = f(x,y)$$

6. 证明：首先 $|f(x)| \le M$，由 $f(x)$ 可积

$\forall \varepsilon > 0$，存在 $[a,b]$ 的一种分割 T：$a = x_0 < x_1 < \cdots < x_n = b$

使 $\sum_{i=1}^{n} w_i \Delta x_i < \varepsilon/2$

$$\left|\int_a^b f(x)\sin px\,dx\right| = \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\sin px\,dx\right|$$

$$= \left|\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left[f(x) - f(x_i) + f(x_i)\right]\sin px\,dx\right|$$

$$\le \sum_{i=1}^{n}\left|\int_{x_{i-1}}^{x_i}(f(x) - f(x_i))\sin px\,dx\right| + |f(x_i)|\left|\int_{x_{i-1}}^{x_i}\sin px\,dx\right|$$

$$\le \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}|f(x) - f(x_i)|\,dx + |f(x_i)|\cdot\left|\frac{1}{p}\cos px\Big|_{x_{i-1}}^{x_i}\right|$$

$$< \sum_{i=1}^{n}\left(\int_{x_{i-1}}^{x_i} w_i\,dx + \frac{2M}{p}\right) = \sum_{i=1}^{n} w_i \Delta x_i + \frac{2nM}{p}$$

令 $p > \frac{4nM}{\varepsilon}$ 时 $\frac{2nM}{p} < \frac{\varepsilon}{2}$

$< \varepsilon$.

③

七. $\forall \varepsilon>0$. 首先 $|f_n'(x)|\le M$

对 $[a,b]$ 作~~等分~~ k 等份，使 $\frac{b-a}{k}\le\frac{\varepsilon}{M}$

小区间分别为 $a=x_0<x_1<\cdots<x_k=b$ $\quad x_i-x_{i-1}=\frac{b-a}{k}$

而 $f_n(x)$ 在 x_i $(i=0,1,\cdots,n)$ 上收敛，$\therefore$ 存在 $N>0$

使 $n>N$ 时 $|f_n(x_i)-f_{n+p}(x_i)|<\varepsilon$

$\forall x\in[a,b]$. 有 $x\in[x_{i-1},x_i]$

$\therefore$ 当 $n>N$ 时时

$$|f_n(x)-f_{n+p}(x)|=|f_n(x)-f_n(x_i)+f_n(x_i)-f_{n+p}(x_i)+f_{n+p}(x_i)-f_{n+p}(x)|$$

$$\le|f_n(x)-f_n(x_i)|+|f_n(x_i)-f_{n+p}(x_i)|+|f_{n+p}(x_i)-f_{n+p}(x)|$$

$$=|f_n'(\xi)||x-x_i|+|f_n(x_i)-f_{n+p}(x_i)|+|f_{n+p}'(\eta)||x-x_i|$$

$$\le M\cdot|x_{i-1}-x_i|+\varepsilon+M|x_{i-1}-x_i|$$

$$=\varepsilon+\varepsilon+\varepsilon=3\varepsilon.$$

证毕.